2. Klausur   I

1. Aufgabe

i) $\displaystyle\int_1^2 \left(4\sqrt[3]{x} + \frac{1}{2\sqrt{x}} - \frac{1}{x}\right) dx = \int_1^2 \left(4x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{x}\right) dx =$

$\displaystyle = \left[3x^{\frac{4}{3}} + x^{\frac{1}{2}} - \ln x\right]_1^2 = 4 2 - 2\sqrt{2} - 2 2 - 3 - 1 = 44 + 2\sqrt{2} - 3\ln 2$

4.74232

ii) $\displaystyle\int_0^1 \cosh x \, dx = \left[\sinh x\right]_0^1 = \sinh 1 = \frac{1}{2}\left(e - \frac{1}{e}\right)$ 1.17520

iii) $\displaystyle\int_0^1 \frac{x^4}{1+x^5} dx = \frac{1}{5}\int_0^1 \frac{5x^4}{1+x^5} dx = \frac{1}{5}\left[\ln(1+x^5)\right]_0^1 = \frac{1}{5}\ln 2$ 0.13863

iv) $\displaystyle\int_1^e \underbrace{x^3}_{f'(x)} \underbrace{\ln x}_{G(x)} dx = \left[\frac{1}{4}x^4 \ln x\right]_1^e - \int_1^e \frac{1}{4}x^4 \frac{1}{x} dx = \frac{1}{4}e^4 - \left[\frac{1}{16}x^4\right]_1^e =$

$\displaystyle = \frac{1}{4}e^4 - \frac{1}{16}e^4 + \frac{1}{16} = \frac{3}{16}e^4 + \frac{1}{16} = \frac{1}{16}(3e^4 + 1)$ 10.29956

v) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x+1} + \sqrt{x}} = \int_0^1 \frac{\sqrt{x+1} - \sqrt{x}}{x+1-x} dx = \int_0^1 \left((x+1)^{\frac{1}{2}} - x^{\frac{1}{2}}\right) dx = \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right]_0^1 =$

$\displaystyle = \frac{2}{3}\left(2\sqrt{2} - 2\right) = \frac{4}{3}\left(\sqrt{2} - 1\right)$ 0.55228

2. Aufgabe

i) $\displaystyle\int_1^c \frac{dx}{x^6} = \left[-\frac{1}{5}\frac{1}{x^5}\right]_1^c = -\frac{1}{5}\frac{1}{c^5} + \frac{1}{5} \to \underline{\underline{\frac{1}{5}}}$ bei $c \to \infty$

ii) $\displaystyle\int_0^c \tan x \, dx = -\int_0^c \frac{-\sin x}{\cos x} dx = -\left[\ln \cos x\right]_0^c = -\ln \cos c \to \infty$ bei $c \uparrow \frac{\pi}{2}$

existiert nicht

iii) $\displaystyle\int_c^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int_{\sqrt{c}}^1 e^u du = 2\left(e - e^{\sqrt{c}}\right) \to \underline{\underline{2(e-1)}}$ bei $c \downarrow 0$

$u = \sqrt{x}$ $du = \frac{1}{2\sqrt{x}} dx$

3.43656

b) $\int_1^c \left(\frac{1}{2} \frac{2x}{x^2+p} - \frac{p}{1+x}\right) dx = \left[\frac{1}{2} \ln(x^2+p) - p \ln(1+x)\right]_1^c =$

$= \left[\ln \frac{\sqrt{x^2+p}}{(x+1)^p}\right]_1^c = \ln \frac{\sqrt{c^2+p}}{(c+1)^p} - \ln \frac{\sqrt{1+p}}{2^p}$

$$\frac{\sqrt{c^2+p}}{(c+1)^p} = \frac{c\sqrt{1+\frac{p}{c}}}{c^p\left(1+\frac{1}{c}\right)^p} = c^{1-p} \frac{\sqrt{1+\frac{p}{c}}}{\left(1+\frac{1}{c}\right)^p} \to \begin{cases} \infty & 0<p<1 \\ 1 & p=1 \\ 0 & 1<p<\infty \end{cases}$$

also $\lim_{c\to\infty} \ln \frac{\sqrt{c^2+p}}{(c+1)^p}$ exist. nur wenn $p=1$, dann $=0$ und

$$\int_1^\infty \ldots = -\ln\frac{\sqrt{2}}{2} = \ln\sqrt{2}.$$

2. Aufgabe a)

i) $\underbrace{\cos x \sinh y}_{A(x,y)} dx + \underbrace{\sin x \cosh y}_{B(x,y)} dy$

$\frac{\partial A}{\partial y}(x,y) = \cos x \cosh y \qquad \frac{\partial B}{\partial x}(x,y) = \cos x \cosh y$

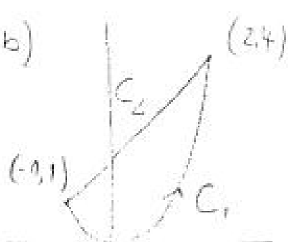
also totales Differential

ii) $\underbrace{(y-x^2)}_{A(x,y)} dx + \underbrace{(y^2-x)}_{B(x,y)} dy$

$\frac{\partial A}{\partial y}(x,y) = 1 \qquad \frac{\partial B}{\partial x}(x,y) = -1$

also kein totales Differential.

b)



C_1: $x(t) = t$, $y(t) = t^2$, $-1 \le t \le 2$; $dx = dt$, $dy = 2t\,dt$

C_2: $x(t) = 2-3t$, $y(t) = 4-3t$, $0 \le t \le 1$; $dx = -3\,dt$, $dy = -3\,dt$

i) C_1 $\int_{-1}^{2} (t^4 - t) 2t\,dt = \int_{-1}^{2} (2t^5 - 2t^2)\,dt = \left[\frac{1}{3}t^6 - \frac{2}{3}t^2\right]_{-1}^{2} =$

$= \frac{64}{3} - \frac{16}{3} - \frac{1}{3} - \frac{2}{3} = +15$

$$C_2 \quad \int_0^1 \Big((-3)\big(4-3t-(4-12t+9t^2)\big) + (-3)\big(16-24t-9t^2-2+3t\big)\Big)\,dt =$$

$$= -2 \int_0^1 (14-12t)\,dt = -3 \left[14t-6t^2\right]_0^1 = -3 \cdot 8 = -24$$

insgesamt $\quad \int_C = -9.$

4. Aufgabe

i) Trapezregel $\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(x_0)+2f(x_1)+ \quad +2f(x_{n-1})+f(x_n)\right)$

$$\int_0^b x^3 dx \approx \frac{b}{8}\left(2\frac{b^3}{4^3}+2\frac{2^3 b^3}{4^3}+2\frac{3^3 b^3}{4^3}+b^3\right)$$

$x_0=0$
$x_1=\frac{b}{4}$
$x_2=2\frac{b}{4}$
$x_3=3\frac{b}{4}$
$x_4=b$

$$=\frac{b^4}{8}\left(\frac{1}{32}+\frac{8}{32}+\frac{27}{32}+\frac{32}{32}\right)=\frac{b^4}{8}\frac{68}{32}=\frac{17}{64}b^4$$

ii) Simpsonregel $\int_a^b f(x)dx \approx \frac{b-a}{3n}\left(f(x_0)+4f(x_1)+2f(x_2)+ \quad +4f(x_{n-1})+f(x_n)\right)$

$$\int_0^b x^3 dx \approx \frac{b}{6}\left(4\frac{b^3}{8}+b^3\right)=\frac{b^4}{6}\left(\frac{1}{2}+1\right)=\frac{b^4}{4} \quad \text{exakt!}$$

$x_0=0$
$x_1=\frac{b}{2}$
$x_2=b$

5. Aufgabe i) $y^2y'=1-2x \qquad y(0)=3$

$$\int y^2 dy=\int(1-2x)dx \qquad \frac{1}{3}y^3=x-x^2+C \qquad y^3=3(x-x^2+C)$$

$$y(x)=\sqrt[3]{3(x-x^2+C)} \qquad 3=y(0)=\sqrt[3]{3C} \qquad C=9$$

ii) $y-xy'=\frac{1}{2}(1+x^2y') \qquad y(1)=1$

$$-xy'-\frac{1}{2}x^2y'=-y+\frac{1}{2} \qquad (2x+x^2)y'=2y-1$$

$$\int\frac{dy}{2y-1}=\int\frac{dx}{x(x+2)}=\frac{1}{2}\int\left(\frac{1}{x}-\frac{1}{x+2}\right)dx$$

$$\frac{1}{2}\ln\left|y-\frac{1}{2}\right|=\frac{1}{2}\left(\ln x-\ln(x+2)\right)+C=\frac{1}{2}\ln\frac{x}{x+2}+C$$

$$\ln\left|y-\frac{1}{2}\right|=\ln\left(\frac{x}{x+2}\right)+2C$$

$$\left|y-\frac{1}{2}\right|=\frac{x}{x+2}e^{2C} \qquad y(x)=\frac{1}{2}+K\frac{x}{x+2} \qquad 1=y(1)=\frac{1}{2}+\frac{K}{3}$$

$$K=\frac{3}{2} \qquad y(x)=\frac{1}{2}+\frac{3}{2}\frac{x}{x+2}$$

6. Aufgabe: $f: \mathbb{R}^2 \to \mathbb{R}$ partiell differenzierbar und $p > 0$ mit

$$x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y}(x,y) = p f(x,y) \qquad \text{für alle } (x,y) \in \mathbb{R}^2.$$

i) $(x,y) \in \mathbb{R}^2$ fest

$h: h(t) = f(tx, ty)$ für alle $t > 0$

erfüllt

$$h'(t) = \frac{\partial f}{\partial x}(tx, ty)x + \frac{\partial f}{\partial y}(tx, ty)y = \frac{1}{t}\left(\frac{\partial f}{\partial x}(tx, ty)tx + \frac{\partial f}{\partial y}(tx, ty)ty\right)$$

$$= \frac{1}{t} p f(tx, ty) = \frac{1}{t} p h(t)$$

ii) $h'(t) = \frac{p}{t} h(t)$ $\qquad \frac{h'(t)}{h(t)} = \frac{p}{t}$ $\qquad \ln|h(t)| = p \ln t + C$

$h(t) = K t^p$; $h(1) = f(x,y)$ also gilt

$h(t) = f(x,y) t^p$ für alle $t > 0$

Wir haben gezeigt $\quad f(tx, ty) = t^p f(x,y)$