Lösungen der Nachklausur

1. Aufgabe

a)

i) $a_n = \frac{n^2}{n+2} - \frac{n^2+1}{n} = \frac{n^2-(n^2+1)(n+2)}{n(n+2)} = -\frac{2n^2+n+2}{n^2+2n} \longrightarrow -2$

ii) $a_n = \left(1-\frac{1}{n}\right)^{\frac{n}{2}} = \left[\left(1-\frac{1}{n}\right)^n\right]^{\frac{1}{2}} \longrightarrow \sqrt{e}$

b) i) $\frac{x^{\frac{1}{5}} - a^{\frac{1}{5}}}{x^{\frac{1}{2}} - a^{\frac{1}{2}}}$, l'Hosp Regel $\frac{\frac{1}{5}x^{-\frac{4}{5}}}{\frac{1}{2}x^{-\frac{1}{2}}} = \frac{2}{5}x^{-\frac{3}{10}} \longrightarrow \frac{2}{5}\frac{1}{a^{3/10}}$ bzw. a

ii) $\frac{2x - \sin x}{3x+\cos x} = \frac{2 - \frac{\sin x}{x}}{3 + \frac{\cos x}{x}} \longrightarrow \frac{2}{3}$

c) $\sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{4}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{\pi}{4}\right)^n = \frac{1}{1+\frac{\pi}{4}} = \frac{4}{4+\pi}$

„geometrische Reihe"

d) $\sum_{n=0}^{\infty} e^{nx} = \sum_{n=0}^{\infty} (e^x)^n$ konvergiert genau wenn $e^x < 1$, also genau wenn $x < 0$

2. Aufgabe

f $f(x,y) = x^3 + y^2 - 6xy - 39x + 18y + 1 \quad \forall (x,y) \in \mathbb{R}^2$

i) $\frac{\partial f}{\partial x}(x,y) = 3x^2 - 6y - 39x \qquad \frac{\partial f}{\partial y}(x,y) = 2y - 6x + 18$

$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x - 39 \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -6 \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 2$

ii) $f(0.01, 0.98) - f(0,1) \approx \frac{\partial f}{\partial x}(0,1)\, 0.01 + \frac{\partial f}{\partial y}(0,1)(-0.02) = 0.06 - 0.4 = -0.36$

$f(0,1) = 1 \qquad f(0.01, 0.98) \approx 0.64$

iii) $3x^2 - 6y - 39x = 0$ & $2y - 6x + 18 = 0 \qquad -2y = -6x + 18$

$x^2 - 2y - 13x = 0 \qquad x^2 - 19x + 18 = 0$

$\left(x - \frac{19}{2}\right)^2 = \frac{361}{4} - 18 = \frac{289}{4} \qquad x_{1,2} = \frac{19}{2} \pm \frac{17}{2} \qquad x_1 = 18 \quad x_2 = 1$

$y = 3x - 9$ $\quad y_1 = 45$ $\quad y_2 = -6$

„kritische Punkte" sind $P_1\,(18/48)$ $\quad P_2\,(1/-6)$

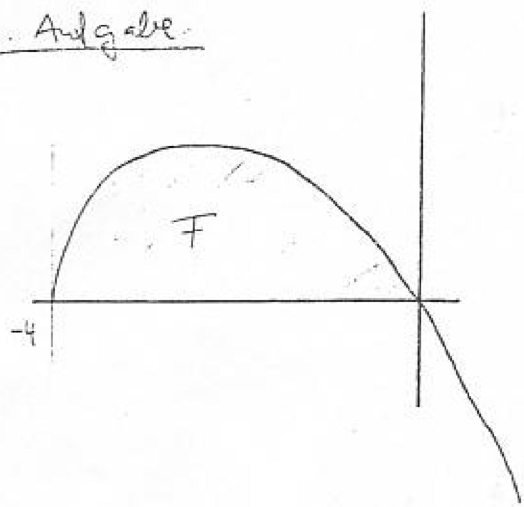
$$D(x,y) := \frac{\partial^2 f}{\partial x^2}(x,y)\,\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 =$$

$$= 2(6x - 39) - 36$$

P_1: $D(18,48) = 102 > 0$, Extremwert , Minimum

P_2: $D(1,-6) = -102 < 0$ kein Extremwert

3. Aufgabe.



$$F = -\frac{1}{2}\int_{-4}^{0} x\sqrt{x+4}\,dx \qquad u = x + 4 \quad x = u - 4$$

$$= -\frac{1}{2}\int_0^4 (u-4)\,u^{1/2}\,du = -\frac{1}{2}\int_0^4 \left(u^{3/2} - 4u^{1/2}\right) du$$

$$= -\frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right]_0^4 = -\frac{1}{2}\left(\frac{64}{5} - \frac{64}{3}\right) =$$

$$= -32\left(\frac{1}{5} - \frac{1}{3}\right) = \frac{64}{15}.$$

$$V = \pi\int_{-4}^{0} f(x)^2\,dx = \frac{\pi}{4}\int_{-4}^{0} x^2(x+4)\,dx = \frac{\pi}{4}\int_{-4}^{0}\left(x^3 + 4x^2\right)dx = \frac{\pi}{4}\left[\frac{1}{4}x^4 + \frac{4}{3}x^3\right]_{-4}^{0}$$

$$= \frac{\pi}{4}\left(-64 + \frac{4}{3}\cdot 64\right) = \frac{16}{3}\pi$$

4. Aufgabe: $\quad$ a)

$$i)\ \int_1^4 \frac{5x^2 - 3x + 4}{\sqrt{x}}\,dx = \int_1^4\left(5x^{3/2} - 3x^{1/2} + 4x^{-1/2}\right)dx = \left[2x^{5/2} - 2x^{3/2} + 8x^{1/2}\right]_1^4 =$$

$$= 64 - 16 + 16 - 2 + 2 - 8 = 56$$

ii) $\int_0^1 \frac{x^5}{1+x^6}\,dx = \frac{1}{6}\int_0^1 \frac{6x^5}{1+x^6}\,dx = \frac{1}{6}\left[\ln(1+x^6)\right]_0^1 = \frac{1}{6}\ln 2$

iii) $\int_1^e \underbrace{x^\alpha}_{F'(x)} \underbrace{\ln x}_{G(x)} = \left[\frac{1}{\alpha+1}x^{\alpha+1}\ln x\right]_1^e - \frac{1}{\alpha+1}\int_1^e x^{\alpha+1}\frac{1}{x}\,dx =$

$= \frac{e^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1}\int_1^e x^\alpha dx = \frac{e^{\alpha+1}}{\alpha+1} - \frac{1}{(\alpha+1)^2}\left[x^{\alpha+1}\right]_1^e =$

$= \frac{e^{\alpha+1}}{\alpha+1} - \frac{1}{(\alpha+1)^2}\left(e^{\alpha+1}-1\right)$

b) i) $\int_1^e \frac{dx}{x\ln x} = \lim_{c\downarrow 1}\int_c^e \frac{dx}{x\ln x} = \lim_{c\downarrow 1}\left[\ln\ln x\right]_c^e = \lim_{c\downarrow 1}\ln\ln c$

existiert nicht!

ii) $\int_0^\infty e^{-\sqrt{x}}dx = \lim_{c\to\infty}\int_0^c e^{-\sqrt{x}}dx = \lim_{c\to\infty}\int_0^{\sqrt{c}} e^{-u}2u\,du =$

$u = \sqrt{x}$
$x = u^2$
$dx = 2u\,du$

$2\int_0^c \underbrace{u}_{G(u)}\underbrace{e^{-u}}_{F'(u)}du = 2\left[-ue^{-u}\right]_0^c + 2\int_0^c e^{-u}du = -2ce^{-c} - 2(e^{-c}-1)$

$\longrightarrow 2$ bei $c\to\infty$

$\int_0^\infty e^{-\sqrt{x}}dx = 2$

c) $\int_0^\infty x e^{-ax^2}dx = \lim_{c\to\infty}\int_0^c x e^{-ax^2}dx = \lim_{c\to\infty}\left[-\frac{1}{2a}e^{-ax^2}\right]_0^c = \frac{1}{2a}$

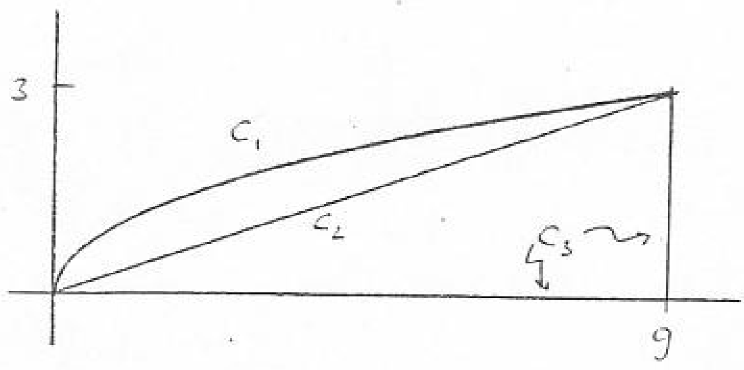
<u>5. Aufgabe</u> $(9+x^2)y' - xy = 0 \qquad y' = \frac{x}{x^2+9}y$

$\int \frac{1}{y}dy = \int \frac{x}{x^2+9}dx \quad \ln|y| = \frac{1}{2}\ln(x^2+9) + K$

$y(x) = C\sqrt{x^2+9}$ i) $y(0) = 3 \quad C = 1 \quad y(x) = \sqrt{x^2+9}$

ii) $y'(1) = 1 \quad C = \frac{1}{\sqrt{10}} \quad y(x) = \sqrt{\frac{x^2+9}{10}}$

6. Aufgabe:



$$\int_C xy^2\,dx + 2x^2y\,dy$$

$$\int_{C_1} xy^2\,dx + 2x^2y\,dy = \int_0^9 \left(t\cdot t + 2t^2\sqrt{t}\,\frac{1}{2\sqrt{t}}\right)dt =$$

C_1: $x(t) = t$, $y(t) = \sqrt{t}$, $0 \le t \le$

$$= \int_0^9 2t^2\,dt = \frac{2}{3}\left[t^3\right]_0^9 = 486$$

$$\int_{C_2} xy^2\,dx + 2x^2y\,dy = \int_0^9 \left(\frac{1}{9}t^3 + \frac{2}{3}t^3\right)dt =$$

C_2: $x(t) = t$, $y(t) = \frac{1}{3}t$, $0 \le t \le 9$

$$= \frac{7}{9}\int_0^9 t^3\,dt = \frac{7}{4\cdot 9}\left[t^4\right]_0^9 = \frac{7}{4}\cdot 9^3 = 1275{,}75$$

$$\int_{C_3} xy^2\,dx + 2x^2y\,dy = \int_{C_{3,1}} + \int_{C_{3,2}}$$

C_3: $C_{3,1} + C_{3,2}$

$C_{3,1}$: $x(t) = t$, $y(t) = 0$, $0 \le t \le 9$

$$\int_{C_{3.1}} = 0 \qquad \int_{C_{3.2}} xy^2\,dx + 2x^2y\,dy =$$

$C_{3.2}$: $x(t) = 9$, $y(t) = t$, $0 \le t \le 3$

$$= \int_0^3 \left(9t^2\cdot 0 + 2\cdot 81\,t\right)dt = 2\cdot 81\left[\frac{1}{2}t^2\right]^3 = 9\cdot 81 = 729.$$