Klausur

1. Aufgabe: a) $n(t) = n_0 e^{\alpha t}$ $\qquad n(0) = 4421$ $\qquad$ jährliche Zunahme: 16%

i) $n_0 = n(0) = 4421$ $\qquad n(t+1) - n(t) = 0.16\, n(t)$

$n_0 e^{\alpha(t+1)} - n_0 e^{\alpha t} = 0.16\, n_0 e^{\alpha t}$ $\qquad e^{\alpha} - 1 = 0.16$ $\qquad e^{\alpha} = 1.16$

$\alpha = \ln 1.16 = 0.1484$

Somit ist $n(t) = 4421\, e^{0.1484 t}$

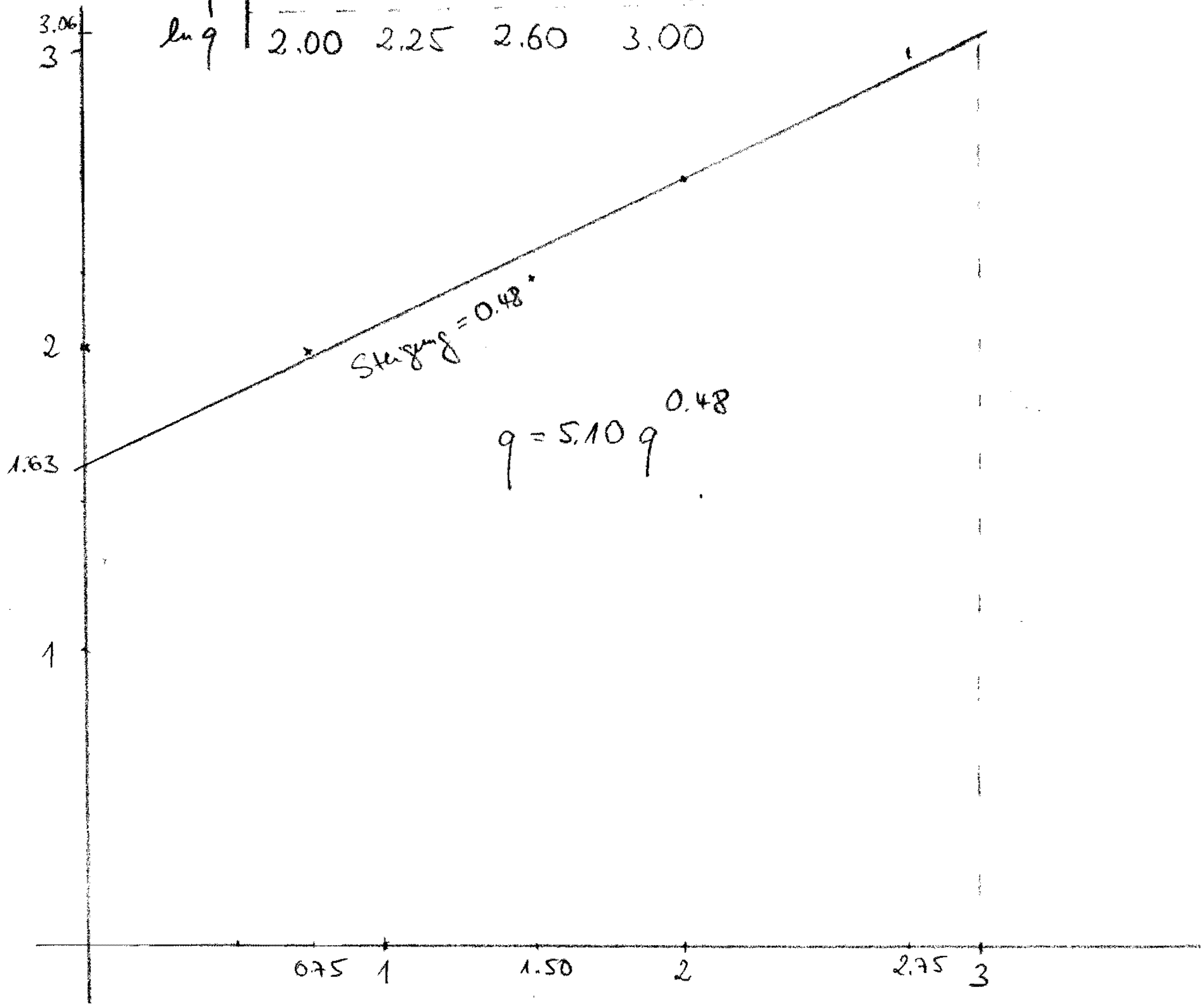
ii) $n(5.5) = 10000.9 \approx 10001$ Tiere

iii) $n(t+T) = 2n(t)$ $\qquad n_0 e^{\alpha(t+T)} = 2 n_0 e^{\alpha t}$ $\qquad e^{\alpha T} = 2$

$\alpha T = \ln 2$ $\qquad T = \dfrac{\ln 2}{\alpha} = 4.670$ Jahre

b) $q = C p^a$ $\qquad \ln q = \ln C + a \ln p$

$\ln p$	0.75	1.50	2.00	2.75
p	2.12	4.48	7.39	15.64
q	7.38	9.49	13.47	20.08
$\ln q$	2.00	2.25	2.60	3.00



ii) (p_1, q_1) und (p_2, q_2) $\quad q_1 = C p_1^a \quad q_2 = C p_2^a$

$$\frac{q_1}{q_2} = \frac{p_1^a}{p_2^a} = \left(\frac{p_1}{p_2}\right)^a \qquad a \ln\left(\frac{p_1}{p_2}\right) = \ln\left(\frac{q_1}{q_2}\right) \qquad a = \frac{\ln q_1 - \ln q_2}{\ln p_1 - \ln p_2}$$

$$a = \frac{2.00 - 3.00}{0.75 - 2.75} = \frac{1}{2} \qquad C = \frac{q_1}{p_1^a} = 5.07 .$$

2. Aufgabe: a)

i) $$\frac{18n^3 - 2n - 17}{6n^3 + 5n^2 + 1} = \frac{18 - \frac{2}{n^2} - \frac{17}{n^3}}{6 + \frac{5}{n} + \frac{1}{n^3}} \to \frac{18}{6} = 3$$

ii) $$\frac{n^2+4}{n+3} - \frac{n^3+2}{n^2+5} = \frac{(n^2+4)(n^2+5) - (n^3+2)(n+3)}{(n+3)(n^2+5)} =$$

$$= \frac{n^4 + 5n^2 + 4n^2 + 20 - (n^4 + 3n^3 + 2n + 6)}{(n+3)(n^2+5)} = \frac{-3n^3 + 9n^2 - 2n + 14}{(n+3)(n^2+5)} \longrightarrow -3$$

bei $n \to \infty$

b) i) $$\sum_{n=0}^{\infty} \frac{e^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{e}{3}\right)^n = \frac{1}{1 - \frac{e}{3}} = \frac{3}{3-e}$$ konvergent als geom. Reihe zu $\frac{e}{3} < 1$.

ii) $\sum_{n=2}^{\infty} \exp\left(\frac{1}{n}\right)$ ist divergent, da $\exp\left(\frac{1}{n}\right) \to 1 \neq 0$ bei $n \to \infty$

c) i) $\sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} (2x)^n$ ist konvergent genau wenn $|2x| < 1$ oder

$|x| < \frac{1}{2}$ (geom. Reihe!), Reihenwert dann $\frac{1}{1-2x} - 1 = \frac{1 - 1 + 2x}{1 - 2x}$

$= \frac{2x}{1-2x}$

ii) $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$ ist konvergent genau wenn $\left|\frac{1}{x}\right| < 1$

oder $|x| > 1$ (geom. Reihe), Reihenwert dann $\frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1}$.

3. Aufgabe: a) $f(x,y) = \sin x \cosh y$ für alle $(x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = \cos x \cosh y \qquad \frac{\partial f}{\partial y}(x,y) = \sin x \sinh y$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin x \cosh y \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = \cos x \sinh y \quad \frac{\partial^2 f}{\partial y^2}(x,y) = \sin x \cosh y$$

b) $f(x,y) = 2x^2 - \frac{1}{4}x^4 + y^2$ für alle $(x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = 4x - x^3 \qquad \frac{\partial f}{\partial y}(x,y) = 2y$$

$$\frac{\partial f}{\partial x}(x,y) = 0 = x(4-x^2) \quad \text{für } x_1 = 0,\ x_2 = 2,\ x_3 = -2$$

$$\frac{\partial f}{\partial y}(x,y) = 0 = 2y \quad \text{für } y = 0$$

drei kritische Punkte: $P_1: (0,0)$ $P_2: (2,0)$ $P_3\,(-2,0)$

Testen ob Extremwerte:

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 4 - 3x^2 \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 2 \quad \frac{\partial^2 f}{\partial x \partial y}(x,y)\ 0$$

$$d(x,y) := \frac{\partial^2 f}{\partial x^2}(x,y)\,\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 = 2(4-3x^2)$$

P_1: $d(0,0) = 8 > 0$ Extremwert! Minimum da $\frac{\partial^2 f}{\partial y^2}(0,0) = 2 > 0$

$P_{2,3}$: $d(\pm 2, 0) = 2(4-12) = -8 < 0$ kein Extremwert!

$f(0,0) = 0$.

4. Aufgabe: a)

i) $$\int_0^1 \left(e^x - 1 - x - \frac{1}{2}x^2\right) dx = \left[e^x - x - \frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^1 =$$

$$= e - 1 - \frac{1}{2} - \frac{1}{6} - 1 = e - \frac{8}{3} \quad (= 0{,}0516)$$

ii) $\int_1^4 \frac{1+\sqrt{x}}{x}\,dx = \int_1^4 \left(\frac{1}{x} + \frac{1}{\sqrt{x}}\right)dx = \left[\ln x + 2\sqrt{x}\right]_1^4 = \ln 4 + 4 - 2 =$

$= 2\ln 2 + 2 \; (= 3.3863)$

iii) $\int_0^{\sqrt{3}} \frac{1}{1+t^2}\,dt = \left[\arctan t\right]_0^{\sqrt{3}} = \arctan\sqrt{3} = \frac{\pi}{3} \; . \; (= 1.0472)$

b) i) $\int_1^\infty \left(\frac{1}{x^3} - \frac{1}{x^4}\right)dx = \lim_{R\to\infty} \int_1^R \left(\frac{1}{x^3} - \frac{1}{x^4}\right)dx$ und

$\int_1^R (x^{-3} - x^{-4})\,dx = \left[-\frac{1}{2}\frac{1}{x^2} + \frac{1}{3}\frac{1}{x^3}\right]_1^R = -\frac{1}{2}\frac{1}{R^2} + \frac{1}{3}\frac{1}{R^3} + \frac{1}{2} - \frac{1}{3} \xrightarrow{\text{bei } R\to\infty} \frac{1}{2} - \frac{1}{3} = \underline{\underline{\frac{1}{6}}}$

ii) $\int_0^1 \frac{1}{2\sqrt{x}} e^{\sqrt{x}}\,dx = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \frac{1}{2\sqrt{x}} e^{\sqrt{x}}\,dx$ und

$\int_\varepsilon^1 \frac{1}{2\sqrt{x}} e^{\sqrt{x}}\,dx = \int_{\sqrt{\varepsilon}}^1 e^u\,du = \left[e^u\right]_{\sqrt{\varepsilon}}^1 = e - e^{\sqrt{\varepsilon}} \to e - 1$ bei $\varepsilon \downarrow 0$.

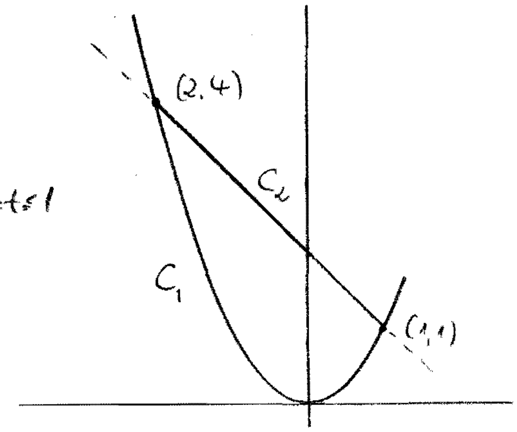
Substituiere
$u = \sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$

iii) $\int_1^\infty x e^{-x}\,dx = \lim_{R\to\infty} \int_1^R x e^{-x}\,dx$ und

$\int_1^R x e^{-x}\,dx = \left[-x e^{-x}\right]_1^R + \int_1^R e^{-x}\,dx = -\frac{R}{e^R} + 1 - \left[e^{-x}\right]_1^R$

$= -\frac{R}{e^R} + \frac{1}{e} - \frac{1}{e^R} + \frac{1}{e} \longrightarrow \underline{\underline{\frac{2}{e}}}$ bei $R \to \infty$

5. Aufgabe:



i) $6y\,dx + 5xy\,dy$

Parameterdarstellung C_1: $x(t)=t,\ y(t)=t^2,\ -2\le t\le 1$

$$\int_{C_1} 6y\,dx + 5xy\,dy = \int_{-2}^{1}\left(6t^2 + 5t\,t^2\,2t\right)dt =$$

$$= \int_{-2}^{1}\left(6t^2+10t^4\right)dt = \left[2t^3+2t^5\right]_{-2}^{1} =$$

$$= 4-(-16-64) = 84$$

Parameterdarstellung C_2: $x(t)=t,\ y(t)=-t+2 \quad -2\le t\le 1$

$$\int_{C_2}\left(6(-t+2)+5t(-t+2)(-1)\right)dt = \int_{-2}^{1}\left(-6t+12+5t^2-10t\right)dt = \int_{-2}^{1}\left(5t^2-16t+12\right)dt$$

$$= \left[\frac{5}{3}t^3-8t^2+12t\right]_{-2}^{1} = \frac{5}{3}-8+12-\left(-\frac{40}{3}-32-24\right) - 15+4+56 = 75$$

ii) $\underbrace{3x^2y}_{A(x,y)}\,dx + \underbrace{(x^3+2y)}_{B(x,y)}\,dy$ Wegen $\frac{\partial A}{\partial y}(x,y) = 3x^2 = \frac{\partial B}{\partial x}(x,y)$

handelt es sich um ein totales Differential, Stammfunktion ist offenbar

$F(x,y) = x^3y + y^2$, somit ist

$$\int_{C_1} 3x^2y\,dx + (x^3+2y)\,dy = \int_{C_2} 3x^2y\,dx + (x^3+2y)\,dy = F(1,1) - F(-2,4) =$$

$$= 1+1-(-32+16) = 18.$$

oder

$$\int_{C_1} 3x^2y\,dx + (x^3+2y)\,dy = \int_{-2}^{1}\left(3t^2t^2 + (t^3+2t^2)2t\right)dt$$

$$= \int_{-2}^{1}\left(3t^4+2t^4+4t^3\right)dt = \int_{-2}^{1}\left(5t^4+4t^3\right)dt = \left[t^5+t^4\right]_{-2}^{1} =$$

$$= 2-(-32+16) = 18$$